A $\frac{3}{2}$-approximation for the Stable Marriage Problem with ties

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The Problem

The Standard Problem

- n men, n women want to marry (each other)
- Each man ranks a subset of the women, each woman ranks a subset of the men
- We want to assign men and women to each other, such that the matching is stable.

More Formally

- Men M, women W.
- Each man m ranks subset $P(m) \subseteq W$ with a linear ordering $\leq_m$.
  A woman w ranks $P(w) \subseteq M$ with $\leq_w$.
- (m, w) is an **acceptable pair** if $w \in P(m)$ and $m \in P(w)$.

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- Each man m ranks subset $P(m) \subseteq W$ with a linear ordering $\leq_m$.
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- (m, w) is an **acceptable pair** if $w \in P(m)$ and $m \in P(w)$.
- A matching is a subset μ of acceptable pairs such that each man and each woman only occur once.
- In a matching μ, we say a man m is **single** if m doesn't occur, otherwise let $\mu(m)$ be m's partner.
  Same for women.

Stability

Definition

A matching μ *is* **not stable** *iff there is an acceptable pair* (m, w) *such that:*

- m *is single or* $\mu(m) <_m w$, *and*
- w *is single or* $\mu(w) <_w m$.

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More positively:

Definition

A matching μ *is* **stable** *iff for each acceptable pair* (m, w), *we have*

- $\exists\,(m, w') \in \mu$ *such that* $w' \geq_m w$, *or*
- $\exists\,(m', w) \in \mu$ *such that* $m' \geq_w m$.

The Solution

Stable marriage exists and can be found easily

The Gale-Shapley algorithm (1962)

- Men propose to women
- Women tentatively accept proposals, become engaged
- If a woman receives two proposals, she rejects the less desirable man

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Algorithm 2 The Gale-Shapley algorithm

```
while there is a single man m who hasn't proposed to all women in P(m) do
    Let w be the most preferred woman in P(m) that m hasn't proposed to
    if w is engaged to a man m' then
        w becomes engaged to max_w(m, m'); w rejects min_w(m, m').
    else
        w becomes engaged to m.
    end if
end while
return μ = {(m, w) : m is engaged to w}.
```

Correctness of the Gale-Shapley algorithm

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Correctness of the Gale-Shapley algorithm

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- *Once a woman becomes engaged, she stays engaged*
- *Furthermore, her partners get more desirable over time*
- *This means that if* m *proposes to* w, *then* w *is married in* μ *and* $\mu(w) \geq_w m$.

Correctness of the Gale-Shapley algorithm

Observation

If m proposes to w, then w is married in μ and $\mu(w) \geq_w m$.

Theorem

The matching μ is stable.

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Theorem

The matching μ is stable.

Proof.

Let (m, w) be an acceptable pair. There are two cases:

1. m proposed to w.
   Then, there is a man m' such that $(m', w) \in \mu$ and $m' \geq_w m$.
2. m didn't propose to w.
   This means that before reaching w, m must have become engaged (till the end) to some woman w'.
   So we have $(m, w') \in \mu$ such that $w' \geq_m w$. □

What about optimal stable matchings?

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- Maximizing the number of married people is boring:

Theorem (Rural Hospital Theorem, Roth 1986)

In any two stable matchings, the set of single people is the same.

- Maximizing an objective function over the set of pairs can be done in polynomial time (R. W. Irving, D. Gusfield 1987; really nice!)

The Variation

Introducing ties

- $\leq_m$ and $\leq_w$ are now weak linear orders
- Write $w \simeq_m w'$ if $w \leq_m w'$ and $w \geq_m w'$.
- Finding the maximum-sized stable matching is NP-complete.

The $\frac{3}{2}$-approximation algorithm

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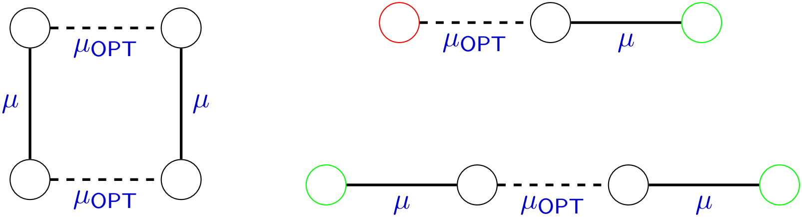
- I slightly simplified it further

What do we need to get a $\frac{3}{2}$-approximation?

Consider the symmetric difference of μ and μ_{OPT} as a graph.



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This can't happen. So μ is always a 2-approximation.

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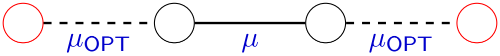
Figure: A 3-augmenting path



If we prevent this from happening, we have a $\frac{3}{2}$-approximation.

The Idea

The Framework

- We will define a set S of "generalized proposals"
- For each man m, let $S_m \subseteq S$ be the proposals involving m. Similarly we define $S_w \subseteq S$ for women.
- For an acceptable pair (m, w), there can be several proposals in $S_m \cap S_w$!

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- For an acceptable pair (m, w), there can be several proposals in $S_m \cap S_w$!
- We will define a linear ordering $\preceq_m$ on S_m and $\preceq_w$ on S_w.
- Execute modified Gale-Shapley to get a "stable set of proposals" $X \subseteq S$.

What do I mean by “stable set”?

- For each man m, X contains at most one element of S_m. Similar for women.
- For each proposal $x \in S_m \cap S_w$, we have
  - $\exists\, y \in X \cap S_m$ such that $y \succeq_m x$, or
  - $\exists\, y \in X \cap S_w$ such that $y \succeq_w x$.

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Once we have X, we "forget" the proposals: We let

$$\mu_X = \{(m, w) \ : \ S_m \cap S_w \cap X \neq \emptyset\}$$

Now, μ_X is a matching, but is it stable???

The Details

The ordering

For each acceptable pair (m, w), S contains a tentative proposal $(m, w, tent)$, a normal proposal $(m, w, norm)$ and a desperate proposal $(m, w, desp)$.

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- Men prefer *tent* and *norm* proposals to *desp* proposals.
- As a tiebreaker, they prefer proposals to more desirable women.
- As a tiebreaker to that, they prefer *tent* to *norm* proposals.
- The remaining ties are broken arbitrarily.

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- Women prefer *desp* and *norm* proposals to *tent* proposals.
- As a tiebreaker, they prefer proposals from more desirable men.
- As a tiebreaker to that, they prefer *desp* to *norm* proposals.
- The remaining ties are broken arbitrarily.

The ordering — illustrated

$$\text{alice} \geq_{\text{bob}} \text{addie} \simeq_{\text{bob}} \text{addy}$$

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$$\frac{(\text{bob}, \text{alice}, \textit{desp})}{(\text{bob}, \text{addie}, \textit{desp})}$$

$$(\text{bob}, \text{addy}, \textit{desp})$$

The ordering — illustrated

alice $\geq_{\text{bob}}$ addie $\simeq_{\text{bob}}$ addy

$S_{\text{bob}}, \preceq_{\text{bob}}$
(bob, alice, *tent*)
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(bob, alice, *desp*)
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$\text{bob} \geq_{\text{alice}} \text{robert} \simeq_{\text{alice}} \text{bob}$

$S_{\text{alice}}, \preceq_{\text{alice}}$
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(bob, alice, *norm*)
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(bob, alice, *tent*)
(robert, alice, *tent*) (rob, alice, *tent*)

Why is μ_X stable?

Observation

- *For a man* m, *if* $(m, w', ?) \succeq_m (m, w, norm)$, *then* $w' \geq_m w$.
- *For woman* w, *if* $(m', w, ?) \succeq_w (m, w, norm)$, *then* $m' \geq_w m$.

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- *For woman* w, *if* $(m', w, ?) \succeq_w (m, w, norm)$, *then* $m' \geq_w m$.

Proof: μ_x is a stable matching.

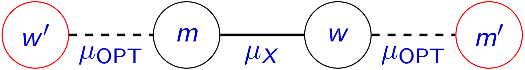
Let (m, w) be an acceptable pair. Define $x = (m, w, norm)$. Since X is a stable set, there are two cases:

- $\exists y \in X \cap S_m$ such that $y \succeq_m x$.
  Choose w' such that $y \in S_{w'}$. So $(m, w') \in \mu_X$. Also, we have $w' \geq_m w$ by the observation.
- $\exists y \in X \cap S_w$ such that $y \succeq_w x$.
  Choose m' such that $y \in S_{m'}$. So $(m', w) \in \mu_X$. Also, we have $m' \geq_w m$ by the observation. □

The Proof of the Approximation Factor

The Setup

Suppose there was a 3-augmenting path:



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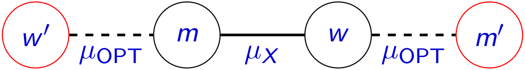
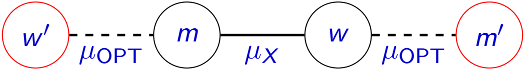


- Choose $x = (m, w, ?)$ such that $x \in X$.
- Since w' is single in μ_X, $S_{w'} \cap X = \emptyset$.
- Since m' is single in μ_X, $S_{m'} \cap X = \emptyset$.

Continuing the Setup



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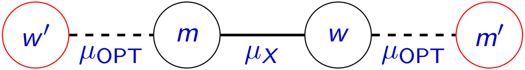

- Since w' is single in μ_X, $S_{w'} \cap X = \emptyset$.
- Since m' is single in μ_X, $S_{m'} \cap X = \emptyset$.
- Since X is a stable set, there must be a proposal $x' \in X \cap S_m$ such that $x' \succeq_m (m, w', tent)$.
- Since X is a stable set, there must be a proposal $x'' \in X \cap S_w$ such that $x'' \succeq_w (m', w, desp)$.

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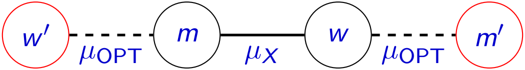

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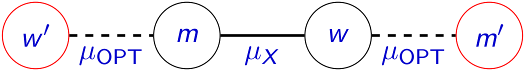
- Since X is a stable set, there must be a proposal $x'' \in X \cap S_w$ such that $x'' \succeq_w (m', w, desp)$.
- Since $X \cap S_m$ has one element, $x' = x$.
- Since $X \cap S_w$ has one element, $x'' = x$.

The Twist



- Since $x = x' = x''$, we have

$$(m, w, ?) \succeq_m (m, w', \mathit{tent}), \quad (m, w, ?) \succeq_w (m', w, \mathit{desp}).$$

- So $?$ can't be *desp*, and $?$ can't be *tent*. So $? = \mathit{norm}$.

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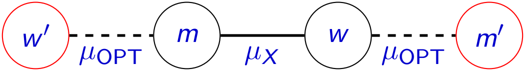

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- From the observation, we see that $w \geq_m w'$.
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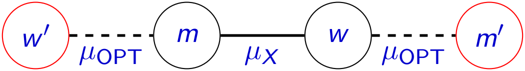

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- So ? can't be *desp*, and ? can't be *tent*. So $? = norm$.
- From the observation, we see that $w \geq_m w'$.
- From the observation, we see that $m \geq_w m'$.
- Looking closely, we can't have $w' \simeq_m w$ or $m' \simeq_w m$.

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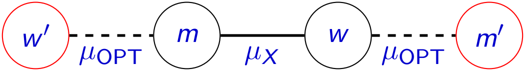

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- So μ_{OPT} is unstable because of (m, w)!

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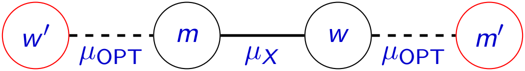

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Other Stuff I Was Doing (not part of the talk)

The Stable Marriage Polytope

AKA "the actual C&O stuff"

- If $\mathcal{A}$ are the acceptable pairs, each matching is a point in $\mathbb{R}^{\mathcal{A}}$.
- The **Stable Marriage Polytope** is the convex hull of all stable marriages.
- Without ties, the SMP can be nicely described by inequalities.
- P. Eirinakis, D. Magos and I. Mourtos (2014) proved (nicely) that the SMP without ties has diameter at most $\frac{n}{2}$.
- Using a similar argument, I proved (disgustingly) that the SMP with ties has diameter at most $\frac{2n}{3}$.

Improving on $\frac{3}{2}$

- Chien-Chung Huang and T. Kavitha (2014) found a $\frac{22}{15} \approx 1.4706$-approximation in the case of one-sided ties.
- They also found a $\frac{10}{7} \approx 1.4286$-approximation for the special case where each tie has length at most two.
- I'm trying to prove that their first algorithm is actually a $\frac{13}{9} \approx 1.4444$-approximation.

Thanks for listening!

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